C^*-ALGEBRAS ASSOCIATED WITH TOPOLOGICAL GROUP QUIVERS II: K-GROUPS

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ABSTRACT. Topological quivers generalize the notion of directed graphs in which the sets of vertices and edges are locally compact (second countable) Hausdorff spaces. Associated to a topological quiver Q is a C^*-correspondence, and in turn, a Cuntz-Pimsner algebra $C^*(Q)$. Given Γ a locally compact group and α and β endomorphisms on Γ, one may construct a topological quiver $Q_{\alpha,\beta}(\Gamma)$ with vertex set Γ, and edge set $\Omega_{\alpha,\beta}(\Gamma) = \{(x,y) \in \Gamma \times \Gamma \,|\, \alpha(y) = \beta(x)\}$. In [52], the author examined the Cuntz-Pimsner algebra $\mathcal{O}_{\alpha,\beta}(\Gamma) := C^*(Q_{\alpha,\beta}(\Gamma))$ and found generators (and their relations) of $\mathcal{O}_{\alpha,\beta}(\Gamma)$. In this paper, the author uses this information to create a six term exact sequence in order to calculate the K-groups of $\mathcal{O}_{\alpha,\beta}(\Gamma)$.

1. Introduction and Notation

1.1. **Background.** Given a quintuple $Q = (X, E, r, s, \lambda)$, where X and E are locally compact (second countable) Hausdorff spaces, r and s are continuous maps from X to E with r open, and $\lambda = \{\lambda_x\}_{x \in E}$ is a system of Radon measures, one can create a corresponding Cuntz-Pimsner C^*-algebra $C^*(Q)$. In [24], Exel, an Huef and Raeburn define C^*-algebras associated with a system (B, α, L) where α is an endomorphism of a unital C^*-algebra B and L is a positive linear map $L : B \to B$ such that $L(\alpha(a)b) = aL(b)$ for all $a, b \in B$ called a *transfer operator*. In fact, the C^*-algebra they generate is a Cuntz-Pimsner algebra and under certain restrictions, a C^*-algebra associated with a topological quiver; in particular, when $B = C(\mathbb{T}^d)$ the continuous function on the d-torus, $F \in M_d(\mathbb{Z})$ and α is the endomorphism

$$\alpha(f)(e^{2\pi i t}) = f(e^{2\pi i F t})$$

for $f \in C(\mathbb{T}^d)$ and $t \in \mathbb{R}^d$. Exel, an Huef and Raeburn then determine a six term exact sequence in which to use to calculate the K-groups of these C^*-algebras. In [52], the author considers a certain class of topological quivers (which extend the notions of Exel, an Huef and Raeburn) $Q = (\Gamma, \Omega_{\alpha,\beta}(\Gamma), r, s, \lambda)$ where Γ is a locally compact group, α and β are endormorphism of Γ,

$$\Omega_{\alpha,\beta}(\Gamma) = \{(x,y) \in \Gamma \times \Gamma \,|\, \alpha(y) = \beta(x)\}$$

and λ is an appropriate family of Radon measures. The resulting Cuntz-Pimsner C^*-algebra, denoted $\mathcal{O}_{\alpha,\beta}(\Gamma)$, was then examined and certain generators and relations where found. We now proceed to generalize the six term exact sequence considered in [24] to C^*-algebras of the form $\mathcal{O}_{\alpha,\beta}(\Gamma)$ where Γ is a compact group.

1.2. **Notation.** The sets of natural numbers, integers, rationals numbers, real numbers and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, respectively. Convention: $\mathbb{N}$ does not contain zero. $\mathbb{Z}_0^+$ will denote the set $\mathbb{N} \cup \{0\}$, $\mathbb{R}^+$ denotes the set $\{r \in \mathbb{R} \,\big|\, r > 0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$. Finally, $\mathbb{Z}_p$ denotes the abelian group $\mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1 \mod p\}$ and $\mathbb{T}$ denotes the torus $\{z \in \mathbb{C} \,\big|\, |z| = 1\}$. Whenever convenient, view $\mathbb{Z}_p \subset \mathbb{T}$ by $\mathbb{Z}_p \cong \{z \in \mathbb{T} \,\big|\, z^p = 1\}$.

For a topological space Y, the closure of Y is denoted $\overline{Y}$. Given a locally compact Hausdorff space X, let

(1) $C(X)$ be the continuous complex functions on X;
(2) $C_b(X)$ be the continuous and bounded complex functions on X;
(3) $C_0(X)$ be the continuous complex functions on X vanishing at infinity;
(4) $C_c(X)$ be the continuous complex functions on X with compact support.

The supremum norm is denoted $||\cdot||_\infty$ and defined by

$$||f||_\infty = \sup_{x \in X}\{|f(x)|\}$$

for each continuous map $f : X \to \mathbb{C}$. For a continuous function $f \in C_c(X)$, denote the open support of f by $\operatorname{osupp} f = \{x \in X \,\big|\, f(x) \neq 0\}$ and the support of f by $\operatorname{supp} f = \overline{\operatorname{osupp} f}$.

For C^*-algebras A and B, A is isomorphic to B will be written $A \cong B$; for example, we use $C(\mathbb{T}^d) \otimes M_N(\mathbb{C}) \cong M_N(C(\mathbb{T}^d))$. Moreover, $A^{\oplus n}$ denotes the n-fold direct sum $A \oplus \cdots \oplus A$. Given a group Γ and a ring R, a normal subgroup, N, of Γ is denoted $N \lhd \Gamma$ and an ideal, I, of R is denoted $I \lhd R$. Note if R is a C^*-algebra then the term ideal denotes a closed two-sided ideal. Furthermore, $\operatorname{End}(\Gamma)$ ($\operatorname{End}(R)$) and $\operatorname{Aut}(\Gamma)$ ($\operatorname{Aut}(R)$) denotes the set of endomorphisms of Γ (R) and automorphisms of Γ (R), respectively. For a map $\gamma : \Gamma \to \operatorname{Aut}(A)$, the *fixed point set* is denoted A^γ and defined by

$$A^\gamma = \{a \in A \,\big|\, \gamma(g)(a) = a \text{ for each } g \in \Gamma\}.$$

Let $\alpha \in C(X)$ then $\alpha^\# \in \operatorname{End}(\mathrm{C(X)})$ denotes the endomorphism of $C(X)$ defined by

$$\alpha^\#(f) = f \circ \alpha \qquad \text{for each } f \in C(X).$$

Let S be a set and define the Kronecker delta function $\delta : S \times S \to \{0, 1\}$ by

$$\delta_s^r := \delta(s, r) = \begin{cases} 0 & \text{if } s \neq r \\ 1 & \text{if } s = r \end{cases}.$$

The set of n by n matrices with coefficients in a set R will be denoted $M_n(R)$ and for any $F \in M_n(R)$, the transpose of F is denoted F^T. Given a function $\sigma : R \to S$, we may create an augmented function $\sigma_n : M_n(R) \to M_n(S)$ via

$$\sigma_n((r_{i,j})_{i,j=1}^n) = (\sigma(r_{i,j}))_{i,j=1}^n$$

for each $(r_{i,j})_{i,j=1}^n \in M_n(R)$. Given vectors $v = (v_1, ..., v_n)$ of length n and $w = (w_1, ..., w_m)$ of length m, denote (v, w) to be the vector $(v, w) = (v_1, ..., v_n, w_1, ..., w_m)$ of length $n + m$.

2. Preliminairies

2.1. **Hilbert C^*-modules.** We begin by defining Hilbert C^*-modules. Further details and references can be found in [48, 63].

Definition 2.1. [48] If A is a C^*-algebra, then a *(right) Hilbert A-module* is a Banach space $\mathcal{E}_A$ together with a right action of A on $\mathcal{E}_A$ and an A-valued inner product $\langle \cdot, \cdot \rangle_A$ satisfying

(1) $\langle \xi, \eta a \rangle_A = \langle \xi, \eta \rangle_A a$
(2) $\langle \xi, \eta \rangle_A = \langle \eta, \xi \rangle_A^*$
(3) $\langle \xi, \xi \rangle \geq 0$ and $||\xi|| = ||\langle \xi, \xi \rangle_A^{1/2}||_A$

for all ξ, $\eta \in \mathcal{E}_A$ and $a \in A$ (if the context is clear, we denote $\mathcal{E}_A$ simply by $\mathcal{E}$). For Hilbert A-modules $\mathcal{E}$ and $\mathcal{F}$, call a function $T : \mathcal{E} \to \mathcal{F}$ *adjointable* if there is a function $T^* : \mathcal{F} \to \mathcal{E}$ such that $\langle T(\xi), \eta \rangle_A = \langle \xi, T^*(\eta) \rangle_A$ for all $\xi \in \mathcal{E}$ and $\eta \in \mathcal{F}$. Let $\mathcal{L}(\mathcal{E}, \mathcal{F})$ denote the set of adjointable (A-linear) operators from $\mathcal{E}$ to $\mathcal{F}$. If $\mathcal{E} = \mathcal{F}$, then $\mathcal{L}(\mathcal{E}) := \mathcal{L}(\mathcal{E}, \mathcal{E})$ is a C^*-algebra (see [48].) Let $\mathcal{K}(\mathcal{E}, \mathcal{F})$ denote the closed two-sided ideal of *compact operators* given by

$$\mathcal{K}(\mathcal{E}, \mathcal{F}) := \overline{\operatorname{span}}\{\theta_{\xi,\eta}^{\mathcal{E},\mathcal{F}} \mid \xi \in \mathcal{E},\, \eta \in \mathcal{F}\}$$

where

$$\theta_{\xi,\eta}^{\mathcal{E},\mathcal{F}}(\zeta) = \xi \langle \eta, \zeta \rangle_A \qquad \text{for each } \zeta \in \mathcal{E}.$$

Similarly, $\mathcal{K}(\mathcal{E}) := \mathcal{K}(\mathcal{E}, \mathcal{E})$ and $\theta_{\xi,\eta}^{\mathcal{E}}$ (or $\theta_{\xi,\eta}$ if understood) denotes $\theta_{\xi,\eta}^{\mathcal{E},\mathcal{E}}$. For Hilbert A-module $\mathcal{E}$, the linear span of $\{\langle \xi, \eta \rangle \mid \xi, \eta \in \mathcal{E}\}$, denoted $\langle \mathcal{E}, \mathcal{E} \rangle$, once closed is a two-sided ideal of A. Note that $\mathcal{E}\langle \mathcal{E}, \mathcal{E} \rangle$ is dense in $\mathcal{E}$ ([48]). The Hilbert module $\mathcal{E}$ is called *full* if $\langle \mathcal{E}, \mathcal{E} \rangle$ is dense in A. The Hilbert module A_A refers to the Hilbert module A over itself, where $\langle a, b \rangle = a^* b$ for all $a, b \in A$.

An *algebraic generating set* for $\mathcal{E}$ is a subset $\{u_i\}_{i \in \mathcal{I}} \subset \mathcal{E}$ for some indexing set $\mathcal{I}$ such that $\mathcal{E}$ equals the linear span of $\{u_i \cdot a \mid i \in \mathcal{I}, a \in A\}$.

Definition 2.2. [37] A subset $\{u_i\}_{i \in \mathcal{I}} \subset \mathcal{E}$ is called a *basis* provided the following reconstruction formula holds for all $\xi \in \mathcal{E}$:

$$\xi = \sum_{i \in \mathcal{I}} u_i \cdot \langle u_i, \xi \rangle \qquad (\text{in } \mathcal{E}, ||\cdot||.)$$

If $\langle u_i, u_j \rangle = \delta_i^j$ as well, call $\{u_i\}_{i \in \mathcal{I}}$ an *orthonormal basis* of $\mathcal{E}$.

Remark 2.3. The preceding definition is in accordance with the finite version in [37], but many other versions exist such as in [24] where $\{u_i\}_{i=1}^n$ is called a finite Parseval frame, or in [68] where this is taken as the definition for *finitely generated.* There has been substantial work done on similar frames (see [32]).

The following notions of C^*-correspondence and morphism may be found in [56, 9, 10, 11, 24, 25, 26, 39]

Definition 2.4. [10, 11] If A and B are C^*-algebras, then an *A–B C^*-correspondence* $\mathcal{E}$ is a right Hilbert B-module $\mathcal{E}_B$ together with a left action of A on $\mathcal{E}$ given by a $*$-homomorphism $\phi_A : A \to \mathcal{L}(\mathcal{E})$, $a \cdot \xi = \phi_A(a)\xi$ for $a \in A$ and $\xi \in \mathcal{E}$. We may

occasionally write, ${}_A\mathcal{E}_B$ to denote an $A - B$ C^*-correspondence and ϕ instead of ϕ_A. Furthermore, if ${}_{A_1}\mathcal{E}_{B_1}$ and ${}_{A_2}\mathcal{F}_{B_2}$ are C^*-correspondences, then a *morphism* $(\pi_1, T, \pi_2) : \mathcal{E} \to \mathcal{F}$ consists of $*$-homomorphisms $\pi_i : A_i \to B_i$ and a linear map $T : \mathcal{E} \to \mathcal{F}$ satisfying

(i) $\pi_2(\langle \xi, \eta\rangle_{A_2}) = \langle T(\xi), T(\eta)\rangle_{B_2}$
(ii) $T(\phi_{A_1}(a_1)\xi) = \phi_{B_1}(\pi_1(a_1))T(\xi)$
(iii) $T(\xi)\pi_2(a_2) = T(\xi a_2)$

for all $\xi, \eta \in \mathcal{E}$ and $a_i \in A_i$.

Notation 2.5. When $A = B$, we refer to ${}_A\mathcal{E}_A$ as a C^*-correspondence over A. For $\mathcal{E}$ a C^*-correspondence over A and $\mathcal{F}$ a C^*-correspondence over B, a morphism $(\pi, T, \pi) : \mathcal{E} \to \mathcal{F}$ will be denoted by (T, π).

Definition 2.6. [56] If $\mathcal{F}$ is the Hilbert module ${}_CC_C$ where C is a C^*-algebra with the inner product $\langle x, y\rangle_B = x^*y$ then call a morphism $(T, \pi) : {}_A\mathcal{E}_B \to C$ of Hilbert modules a *representation* of ${}_A\mathcal{E}_B$ into C.

Remark 2.7. Note that a representation of ${}_A\mathcal{E}_B$ need only satisfying (i) and (ii) of definition 2.4 as it was unnecessary to require (iii). For a proof, see [52, Remark 2.7].

A morphism of Hilbert modules $(T, \pi) : \mathcal{E} \to \mathcal{F}$ yields a $*$-homomorphism $\Psi_T : \mathcal{K}(\mathcal{E}) \to \mathcal{K}(\mathcal{F})$ by
$$\Psi_T(\theta^{\mathcal{E}}_{\xi,\eta}) = \theta^{\mathcal{F}}_{T(\xi),T(\eta)}$$
for $\xi, \eta \in \mathcal{E}$ and if $(S, \sigma) : \mathcal{D} \to \mathcal{E}$, and $(T, \pi) : \mathcal{E} \to \mathcal{F}$ are morphisms of Hilbert modules then $\Psi_T \circ \Psi_S = \Psi_{T\circ S}$. In the case where $\mathcal{F} = B$ a C^*-algebra, we may first identify $\mathcal{K}(B)$ as B, and a representation (T, π) of $\mathcal{E}$ in a C^*-algebra B yields a $*$-homomorphism $\Psi_T : \mathcal{K}(\mathcal{E}) \to B$ given by
$$\Psi_T(\theta_{\xi,\eta}) = T(\xi)T(\eta)^*.$$

Definition 2.8. [56] For a C^*-correspondence $\mathcal{E}$ over A, denote the ideal $\phi^{-1}(\mathcal{K}(\mathcal{E}))$ of A by $J(\mathcal{E})$, and let $J_{\mathcal{E}} = J(\mathcal{E}) \cap (\ker\phi)^{\perp}$ where $(\ker\phi)^{\perp}$ is the ideal $\{a \in A \,\big|\, ab = 0 \text{ for all } b \in \ker\phi\}$. If ${}_A\mathcal{E}_A$ and ${}_B\mathcal{F}_B$ are C^*-correspondences over A and B respectively and $K \lhd J(\mathcal{E})$, a morphism $(T, \pi) : \mathcal{E} \to \mathcal{F}$ is called *coisometric on* K if
$$\Psi_T(\phi_A(a)) = \phi_B(\pi(a))$$
for all $a \in K$, or just *coisometric*, if $K = J(\mathcal{E})$.

Notation 2.9. We denote $C^*(T, \pi)$ to be the C^*-algebra generated by $T(\mathcal{E})$ and $\pi(A)$ where $(T, \pi) : \mathcal{E} \to B$ is a representation of ${}_A\mathcal{E}_A$ in a C^*-algebra B. Furthermore, if $\rho : B \to C$ is a $*$-homomorphism of C^*-algebras, then $\rho \circ (T, \pi)$ denotes the representation $(\rho \circ T, \rho \circ \pi)$ of $\mathcal{E}$.

Definition 2.10. [56] A morphism $(T_{\mathcal{E}}, \pi_{\mathcal{E}})$ coisometric on an ideal K is said to be *universal* if whenever $(T, \pi) : \mathcal{E} \to B$ is a representation coisometric on K, there exists a $*$-homomorphism $\rho : C^*(T_{\mathcal{E}}, \pi_{\mathcal{E}}) \to B$ with $(T, \pi) = \rho \circ (T_{\mathcal{E}}, \pi_{\mathcal{E}})$. The universal C^*-algebra $C^*(T_{\mathcal{E}}, \pi_{\mathcal{E}})$ is called the *relative Cuntz-Pimsner algebra* of $\mathcal{E}$

determined by the ideal K and denoted by $\mathcal{O}(K,\mathcal{E})$. If $K=0$, then $\mathcal{O}(K,\mathcal{E})$ is denoted by $\mathcal{T}(\mathcal{E})$ and called the *universal Toeplitz C^*-algebra* for $\mathcal{E}$. We denote $\mathcal{O}(J_{\mathcal{E}},\mathcal{E})$ by $\mathcal{O}_{\mathcal{E}}$.

2.2. **Topological Quivers.**

Definition 2.11. [56] A *topological quiver* (or *topological directed graph*) $Q = (X,E,Y,r,s,\lambda)$ is a diagram

$$X \xleftarrow{\ s\ } E \xrightarrow{\ r\ } Y$$

where $X,E,$ and Y are second countable locally compact Hausdorff spaces, r and s are continuous maps with r open, along with a family $\lambda=\{\lambda_y | y\in Y\}$ of Radon measures on E satisfying

(1) supp $\lambda_y = r^{-1}(y)$ for all $y\in Y$, and
(2) $y\mapsto \lambda_y(f)=\int_E f(\alpha)d\lambda_y(\alpha)\in C_c(Y)$ for $f\in C_c(E)$.

Remark 2.12. If $X=Y$ then write $Q=(X,E,r,s,\lambda)$ in lieu of (X,E,X,r,s,λ).

Remark 2.13. The author provides a broad history and a series of examples of topological quivers in [51, 52].

Given a topological quiver $Q=(X,E,Y,r,s,\lambda)$, one may associate a correspondence $\mathcal{E}_Q$ of the C^*-algebra $C_0(X)$ to the C^*-algebra $C_0(Y)$. Define left and right actions

$$(a\cdot\xi\cdot b)(e)=a(s(e))\xi(e)b(r(e))$$

by $C_0(X)$ and $C_0(Y)$ respectively on $C_c(E)$. Furthermore, define the $C_c(Y)$-valued inner product

$$\langle\xi,\eta\rangle(y)=\int_{r^{-1}(y)}\overline{\xi(\alpha)}\eta(\alpha)d\lambda_y(\alpha)$$

for $\xi,\eta\in C_c(E)$, $y\in Y$, and let $\mathcal{E}_Q$ be the completion of $C_c(E)$ with respect to the norm

$$||\xi||=||\langle\xi,\xi\rangle^{1/2}||_\infty=||\lambda_y(|\xi|^2)||_\infty^{1/2}.$$

Definition 2.14. Given topological quiver Q over a space X, define the C^*-algebra, $C^*(Q)$ associated with Q to be the Cuntz-Pimnser C^*-algebra $\mathcal{O}_{\mathcal{E}_Q}$ of the correspondence $\mathcal{E}_Q$ over $A=C_0(X)$.

2.3. **Topological Group Quivers.**

Definition 2.15. Let Γ be a (second countable) locally compact group and let $\alpha,\beta\in\mathrm{End}(\Gamma)$ be continuous. Define the closed subgroup, $\Omega_{\alpha,\beta}(\Gamma)$, of $\Gamma\times\Gamma$,

$$\Omega_{\alpha,\beta}(\Gamma)=\{(x,y)\in\Gamma\times\Gamma \,\big|\, \alpha(y)=\beta(x)\}$$

and let $Q_{\alpha,\beta}(\Gamma)=(\Gamma,\Omega_{\alpha,\beta}(\Gamma),r,s,\lambda)$ where r and s are the group homomorphisms defined by

$$r(x,y)=x \qquad \text{and} \qquad s(x,y)=y$$

for each $(x,y)\in\Omega_{\alpha,\beta}(\Gamma)$ and λ_x for $x\in\Gamma$ is the measure on

$$r^{-1}(x)=\{x\}\times\alpha^{-1}(\beta(x))$$

defined by

$$\lambda_x(B) = \mu(y^{-1}s(B \cap r^{-1}(x)) \cap \ker\alpha) \qquad (\text{for any } y \in \alpha^{-1}(\beta(x)))$$

for each measurable $B \subseteq \Omega_{\alpha,\beta}(\Gamma)$ where μ is a left Haar measure (normalized if possible) on $r^{-1}(1_\Gamma) = \{1\} \times \ker\alpha$ (a closed normal subgroup of $\Gamma \times \Gamma$; hence, a locally compact group). Note if $r^{-1}(x) = \emptyset$ then $\alpha^{-1}(\beta(x)) = \emptyset$ and so $\lambda_x = 0$. This measure is well-defined,

$$\text{supp } \lambda_x = \{x\} \times y \ker\alpha = \{x\} \times \alpha^{-1}(\beta(x)) = r^{-1}(x)$$

and $y \mapsto \lambda_y(f)$ is a continuous compactly supported function (cf. [52, Definition 3.1].

Call $Q_{\alpha,\beta}(\Gamma)$ a *topological group relation.* Define $\mathcal{E}_{\alpha,\beta}(\Gamma)$ to be the $C_0(\Gamma)$-correspondence $\mathcal{E}_{Q_{\alpha,\beta}(\Gamma)}$ and form the Cuntz-Pimsner algebra

$$\mathcal{O}_{\alpha,\beta}(\Gamma) := C^*(Q_{\alpha,\beta}(\Gamma)) = \mathcal{O}(J_{\mathcal{E}_{\alpha,\beta}(\Gamma)}, \mathcal{E}_{\alpha,\beta}(\Gamma))$$

and the Toeplitz-Pimsner algebra

$$\mathcal{T}_{\alpha,\beta}(\Gamma) := \mathcal{T}(Q_{\alpha,\beta}(\Gamma)).$$

Remark 2.16. It will be implicitly assumed that Γ is second countable. Furthermore, since Γ is locally compact Hausdorff, $r^{-1}(x)$ is closed and locally compact. Moreover, whenever r is a local homeomorphism, $r^{-1}(x)$ is discrete and hence, λ_x is counting measure (normalized when $|\ker\alpha| < \infty$.)

Example 2.17 ([52])**.** For the compact abelian group $\mathbb{T}^d$, note $\text{End}(\mathbb{T}^d) \cong M_d(\mathbb{Z})$ ([67]); that is, an element $\sigma \in \text{End}(\mathbb{T}^d)$ is of the form σ_F for some $F \in M_d(\mathbb{Z})$ where

$$\sigma_F(e^{2\pi i t}) = e^{2\pi i F t}$$

for each $t \in \mathbb{Z}^d$. To simplify notation, use F and G in place of σ_F and σ_G whenever convenient. For instance,

$$Q_{F,G}(\mathbb{T}^d) := Q_{\sigma_F,\sigma_G}(\mathbb{T}^d)$$

and the C^*-correspondence

$$\mathcal{E}_{F,G}(\mathbb{T}^d) := \mathcal{E}_{\sigma_F,\sigma_G}(\mathbb{T}^d)$$

where $F, G \in M_d(\mathbb{Z})$. We will consider the cases when these maps are surjective; that is, $\det F$ and $\det G$ are non-zero.

Let $F, G \in M_d(\mathbb{Z})$ where $\det F, \det G \neq 0$. Then $|\ker\sigma_F| = |\det F|$ and so, the $C(\mathbb{T}^d)$-valued inner product becomes

$$\langle \xi, \eta \rangle(x) = \frac{1}{|\det F|} \sum_{\sigma_F(y) = \sigma_G(x)} \overline{\xi(x,y)}\eta(x,y)$$

for $\xi, \eta \in \mathcal{E}_{F,G}(\mathbb{T}^d)$ and $x \in \mathbb{T}^d$. This is a finite sum since the number of solutions, y, to $\sigma_F(y) = \sigma_G(x)$ given any $x \in \mathbb{T}^d$ is $|\det F| < \infty$.

Remark 2.18. The left action, ϕ, is defined by

$$\phi(a)\xi(x,y) = a(y)\xi(x,y)$$

for $a \in C(\mathbb{T}^d)$, $\xi \in C(\Omega_{F,G}(\mathbb{T}^d))$ and $(x,y) \in \Omega_{F,G}(\mathbb{T}^d)$. Note: ϕ is injective. To see this claim, let $a \in C(\mathbb{T}^d)$ and assume $\phi(a)\xi = 0$ for each $\xi \in C(\Omega_{F,G}(\mathbb{T}^d))$. Then $a(y)\xi(x,y) = 0$ for each $(x,y) \in \Omega_{F,G}(\mathbb{T}^d)$ and $\xi \in C(\Omega_{F,G}(\mathbb{T}^d))$. Since $s(\Omega_{F,G}(\mathbb{T}^d)) = \{y \in \mathbb{T}^d \,|\, (x,y) \in \Omega_{F,G}(\mathbb{T}^d)\} = \mathbb{T}^d$ by the surjectivity of σ_F, $a = 0$.

Remark 2.19. It was shown in [52] that one may assume the matrix F is positive diagonal.

Let $F = \mathrm{Diag}(a_1, ..., a_d) \in M_d(\mathbb{Z}), G = (b_{jk})_{j,k=1}^d \in M_d(\mathbb{Z})$ where $a_j > 0$ for each $j = 1, ..., d$, $\det G \neq 0$ and let G_j denote the j-th row of G, $(b_{jk})_{k=1}^d$. Further, let $N = \det F = \prod_{j=1}^d a_j > 0$ and let

$$\mathfrak{I}(F) = \{\nu = (\nu_j)_{j=1}^d \in \mathbb{Z}^d \,\big|\, 0 \leq \nu_j \leq a_j - 1\}.$$

The $C(\mathbb{T}^d)$-valued inner product becomes

$$\langle \xi, \eta \rangle(x) = \frac{1}{N} \sum_{\sigma_F(y)=\sigma_G(x)} \overline{\xi(x,y)}\eta(x,y)$$

for all $\xi, \eta \in C(\Omega_{F,G}(\mathbb{T}^d))$ and $x \in \mathbb{T}^d$.

Given $\nu \in \mathfrak{I}(F)$, define $u_\nu \in C(\Omega_{F,G}(\mathbb{T}^d))$ by

$$u_\nu(x,y) = y^\nu = \prod_{j=1}^d y^{\nu_j}$$

for $(x,y) \in \Omega_{F,G}(\mathbb{T}^d)$. It was shown in [52] that $\{u_\nu\}_{\nu \in \mathfrak{I}(F)}$ is a basis for $\mathcal{E}_{F,G}(\mathbb{T}^d)$ and also the following:

Theorem 2.20. [52, Theorem 3.23] Let $F = \mathrm{Diag}(a_1, ..., a_d), G \in M_d(\mathbb{Z})$ where $\det F, \det G \neq 0$ and let G_j be the j-th row vector of G. Further, let $\mathfrak{I}(F)$ denote the set $\{\nu = (\nu_j)_{j=1}^d \in \mathbb{Z}^d \,\big|\, 0 \leq \nu_j \leq a_j - 1\}$. Then $\mathcal{O}_{F,G}(\mathbb{T}^d)$ is the universal C^*-algebra generated by isometries $\{S_\nu\}_{\nu \in \mathfrak{I}(F)}$ and (full spectrum) commuting unitaries $\{U_j\}_{j=1}^d$ that satisfy the relations

(1) $S_\nu^* S_{\nu'} = \langle u_\nu, u_{\nu'} \rangle = \delta_\nu^{\nu'}$,
(2) $U^\nu S = S_\nu$ for all $\nu \in \mathfrak{I}(F)$,
(3) $U_j^{a_j} S = S U^{G_j}$, for all $j = 1, ..., d$ and
(4) $1 = \sum_{\nu \in \mathfrak{I}(F)} S_\nu S_\nu^* = \sum_{\nu \in \mathfrak{I}(F)} U^\nu S S^* U^{-\nu}$

where U^ν denotes $\prod_{j=1}^d U_j^{\nu_j}$. Furthermore, $\mathcal{T}_{\alpha,\beta}(\Gamma)$ is the universal C^*-algebra generated by isometries $\{S_\nu\}_{\nu \in \mathfrak{I}(F)}$ and commuting unitaries $\{U_j\}_{j=1}^d$ that satisfy relations (1)-(3)

3. Six Term Exact Sequence for $\mathcal{O}_{\alpha,\beta}(\Gamma)$

In this section, we follow and extend the approach of [24] to create a six term exact sequence. Let Γ be a compact group with $\alpha, \beta \in \mathrm{End}(\Gamma)$. Suppose the left action for the correspondence, ϕ, is injective where ϕ is defined by

$$\phi(a)\xi(x,y) = a(y)\xi(x,y)$$

for $a \in C(\Gamma)$, $\xi \in C(\Omega_{\alpha,\beta}(\Gamma))$ and $(x,y) \in \Omega_{\alpha,\beta}(\Gamma)$. Furthermore, we shall assume the existence of an orthonormal basis (see Defintion 2.2) $\{u_i\}_{i=0}^{N-1}$ for $\mathcal{E}_{\alpha,\beta}(\Gamma)$.

In order to construct our exact sequence for $K_*(\mathcal{O}_{\alpha,\beta}(\Gamma))$, note the short exact sequence

$$0 \longrightarrow \ker q \xrightarrow{\iota} \mathcal{T}_{\alpha,\beta}(\Gamma) \xrightarrow{q} \mathcal{O}_{\alpha,\beta}(\Gamma) \longrightarrow 0,$$

where $q : \mathcal{T}_{\alpha,\beta}(\Gamma) \to \mathcal{O}_{\alpha,\beta}(\Gamma)$ is the canonical quotient map and $\iota : \ker q \to \mathcal{T}_{\alpha,\beta}(\Gamma)$ is the inclusion homomorphism, induces the six-term exact sequence of K-groups (see [65])

$$\begin{tikzcd}
K_0(\ker\ q) \arrow[r, "\iota_*"] & K_0(\mathcal{T}_{\alpha,\beta}(\Gamma)) \arrow[r, "q_*"] & K_0(\mathcal{O}_{\alpha,\beta}(\Gamma)) \arrow[d, "\delta_0"] \\
K_1(\mathcal{O}_{\alpha,\beta}(\Gamma)) \arrow[u, "\delta_1"] & K_1(\mathcal{T}_{\alpha,\beta}(\Gamma)) \arrow[l, "q_*"] & K_1(\ker\ q) \arrow[l, "\iota_*"]
\end{tikzcd} \tag{3.1}$$

Let $(T, \tilde{\pi})$ denote the universal Toeplitz representation on $\mathcal{E}_{\alpha,\beta}(\Gamma)$; that is, $\pi = q \circ \tilde{\pi}$ is the morphism $C(\Gamma) \to \mathcal{O}_{\alpha,\beta}(\Gamma)$. As shown in [60, Theorem 4.4], the homomorphism $\tilde{\pi} : C(\Gamma) \to \mathcal{T}_{\alpha,\beta}(\Gamma)$ induces an isomorphism of $K_i(C(\Gamma))$ onto $K_i(\mathcal{T}_{\alpha,\beta}(\Gamma))$. Thus we may replace $K_i(\mathcal{T}_{\alpha,\beta}(\Gamma))$ with $K_i(C(\Gamma))$ provided we can identify the resulting maps. We intend to show that (3.1) induces the six-term exact sequence

$$\begin{tikzcd}
K_0(C(\Gamma)) \arrow[r, "1-\Omega_*"] & K_0(C(\Gamma)) \arrow[r, "\pi_*"] & K_0(\mathcal{O}_{\alpha,\beta}(\Gamma)) \arrow[d] \\
K_1(\mathcal{O}_{\alpha,\beta}(\Gamma)) \arrow[u] & K_1(C(\Gamma)) \arrow[l, "\pi_*"] & K_1(C(\Gamma)) \arrow[l, "1-\Omega_*"]
\end{tikzcd} \tag{3.2}$$

for $\pi = q \circ \tilde{\pi} : C(\Gamma) \to \mathcal{O}_{\alpha,\beta}(\Gamma)$ and an appropriately chosen homomorphism $\Omega : C(\Gamma) \to M_N(C(\Gamma))$.

Lemma 3.1. *Define $\Omega : C(\Gamma) \to M_N(C(\Gamma))$ by $\Omega(a) = (\langle u_i, a \cdot u_j\rangle)_{i,j=0}^{N-1}$. Then Ω is a unital homomorphism and $\Omega(\alpha^{\#}(a))$ is the diagonal matrix $\beta^{\#}(a)1_N$ for all $a \in C(\Gamma)$.*

Proof. Let $a, b \in C(\Gamma)$. Then the (i,j)-entry of $\Omega(a)\Omega(b)$ is

$$\begin{aligned}(\Omega(a)\Omega(b))_{i,j} &= \sum_{k=0}^{N-1} \langle u_i, a \cdot u_k\rangle\langle u_k, b \cdot u_j\rangle \\ &= \langle u_i, a \cdot (\sum_k u_k \cdot \langle u_k, b\cdot u_j\rangle)\rangle \\ &= \langle u_i, a \cdot (b \cdot u_j)\rangle \\ &= \Omega(ab)_{i,j}.\end{aligned}$$

Furthermore, for a^* denoting the map $a^*(x) = \overline{a(x)}$ for $x \in \Gamma$,

$$\Omega(a^*) = (\langle u_i, a^* \cdot u_j\rangle)_{i,j} = (\langle a \cdot u_i, u_j\rangle)_{i,j} = (\langle u_j, a \cdot u_i\rangle^*)_{i,j} = \Omega(a)^*$$

and

$$\Omega(1) = (\langle u_i, u_j\rangle)_{i,j} = (\delta_i^j)_{i,j} = 1_N.$$

Finally, let $x \in \Gamma$. Then

$$\begin{aligned}\Omega(\alpha^\#(a))_{i,j}(x) &= \langle u_i, \alpha^\#(a) \cdot u_j\rangle(x) \\ &= \int_{r^{-1}(x)} \overline{u_i(e)}a(\alpha(s(e)))u_j(e)\, d\lambda_x(e) \\ &= \int_{r^{-1}(x)} \overline{u_i(e)}a(\beta(x))u_j(e)\, d\lambda_x(e) \\ &= a(\beta(x)) \int_{r^{-1}(x)} \overline{u_i(e)}u_j(e)\, d\lambda_x(e) \\ &= a(\beta(x))\langle u_i, u_j\rangle(x) \\ &= \delta_i^j \beta^\#(a)(x);\end{aligned}$$

hence, $\Omega(\alpha^\#(a)) = \beta^\#(a)1_N$. □

In order to describe $\ker q$, use the notation $\mathcal{E}^{\otimes k} := \mathcal{E}_{\alpha,\beta}(\Gamma)^{\otimes k}$ for the k-fold internal tensor product of C^*-correspondences ([48]) $\mathcal{E}_{\alpha,\beta}(\Gamma) \otimes \cdots \otimes \mathcal{E}_{\alpha,\beta}(\Gamma)$, which is itself a C^*-correspondence over $A = C(\Gamma)$. For the universal covariant representation $(T, \tilde{\pi}) : \mathcal{E}_{\alpha,\beta}(\Gamma) \to \mathcal{T}_{\alpha,\beta}(\Gamma)$ (that is, $q(T_j)$ is the isometry S_j with $T(u_j) = T_j$,) there is, in fact, a Toeplitz representation $(T^{\otimes k}, \tilde{\pi})$ of $\mathcal{T}_{\alpha,\beta}(\Gamma)$ such that $T^{\otimes k}(\xi) = \prod_{i=1}^k T(\xi_i)$ for all elementary tensors $\xi = \xi_1 \otimes \cdots \otimes \xi_k$ where $\xi_i \in \mathcal{E}_{\alpha,\beta}(\Gamma)$ (see [27, Proposition 1.8] where the term "Hilbert bimodule" is used instead of C^*-correspondence.) Note $\mathcal{E}_{\alpha,\beta}(\Gamma)^{\otimes 0} := C(\Gamma)$ and $T^{\otimes 0} := \tilde{\pi}$. By [27, Lemma 2.4],

$$\mathcal{T}_{\alpha,\beta}(\Gamma) = \overline{\operatorname{span}}\{T^{\otimes k}(\xi)T^{\otimes k'}(\eta)^* \,\big|\, k, k' \geq 0, \xi \in \mathcal{E}^{\otimes k}, \eta \in \mathcal{E}^{\otimes k'}\}.$$

Next, let $p = \sum_{j=0}^{N-1} T_j T_j^*$. The proceeding lemmas and propositions are essentially those found in [24, Lemma 3.2, Lemma 3.3 & Proposition 3.4] with some changes.

Lemma 3.2. With the preceding notation:

(1) p is a projection which commutes with $\tilde{\pi}(a)$ for all $a \in C(\Gamma)$

(2) $1-p$ is a full projection in $\ker q$
(3) $(1-p)T^{\otimes k}(\xi)=0$ for all $\xi\in\mathcal{E}^{\otimes k}$ and $k\geq 1$
(4) $\ker q=\overline{\operatorname{span}}\{T^{\otimes k}(\xi)(1-p)T^{\otimes k'}(\eta)^* \,\big|\, k,k'\geq 0, \xi\in\mathcal{E}^{\otimes k}, \eta\in\mathcal{E}^{\otimes k'}\}$

Proof. (1) Recall that $T_i^*T_j=\tilde{\pi}(\langle u_i,u_j\rangle)=\delta_i^j$. Thus, $p^2=p$ and $p^*=p$. Furthermore,

$$\begin{aligned} p\tilde{\pi}(a)p &= \sum_{j,k=1}^{N-1} T_jT_j^*\tilde{\pi}(a)T_kT_k^* = \sum_{j,k=0}^{N-1} T_j\tilde{\pi}(\langle u_j, a\cdot u_k\rangle)T_k^* \\ &= \sum_{j,k=0}^{N-1} T(u_j\langle u_j, a\cdot u_k\rangle)T_k^* = \sum_{k=0}^{N-1} T(\sum_{j=0}^{N-1} u_j\langle u_j, a\cdot u_k\rangle)T_k^* \\ &= \sum_{k=0}^{N-1} T(a\cdot u_k)T_k^* = \tilde{\pi}(a)p \end{aligned}$$

and so,

$$p\tilde{\pi}(a)=(\tilde{\pi}(a)^*p)^*=(p\tilde{\pi}(a)^*p)^*=p\tilde{\pi}(a)p=\tilde{\pi}(a)p.$$

(2) Recall $\phi(a)=\sum_{j=0}^{N-1}\theta_{a\cdot u_j,u_j}$, so

$$\Psi_T(\phi(a))=\sum_{j=0}^{N-1} T(a\cdot u_j)T(u_j)^*=\tilde{\pi}(a)p$$

and

$$q(1-p)=q(\tilde{\pi}(1)-\tilde{\pi}(1)p)=q(\tilde{\pi}(1)-\Psi_T(\phi(1)))=0.$$

Hence, $1-p=\tilde{\pi}(1)-\tilde{\pi}(1)p\in\ker q$ and since $\ker q$ is the ideal in $\mathcal{T}_{\alpha,\beta}(\Gamma)$ generated by $\{\tilde{\pi}(a)-\Psi_T(\phi(a)) \,\big|\, a\in C(\Gamma)\}$ and $1-p\in\ker q$, $\ker q$ is the ideal generated by $\{\tilde{\pi}(a)(1-p) \,\big|\, a\in C(\Gamma)\}$. Hence $1-p$ is full.

(3) Let $\xi\in\mathcal{E}_{\alpha,\beta}(\Gamma)$ then

$$pT(\xi)=\sum_{j=0}^{N-1} T(u_j)T(u_j)^*T(\xi)=\sum_{j=0}^{N-1} T(u_j\langle u_j,\xi\rangle)=T(\xi).$$

Thus, $(1-p)T(\xi)=0$. Now for $k>1$, let $\xi=\xi_1\otimes ...\otimes\xi_k$. Then

$$(1-p)T^{\otimes k}(\xi)=(1-p)\prod_{j=0}^{k} T(\xi_j)=0$$

and hence, by linearity and continuity, (3) has been proven.

(4) Since $\ker q=\mathcal{T}_{\alpha,\beta}(\Gamma)(1-p)\mathcal{T}_{\alpha,\beta}(\Gamma)$, the description of $\mathcal{T}_{\alpha,\beta}(\Gamma)$ preceding Lemma 3.2 paired with (3) gives the desired result.

□

Lemma 3.3. There exists a homomorphism $\rho: C(\Gamma)\to\ker q\subset\mathcal{T}_{\alpha,\beta}(\Gamma)$ such that $\rho(a)=\tilde{\pi}(a)(1-p)$ and ρ is an isomorphism of $C(\Gamma)$ onto the full corner C^*-algebra

$(1-p)\ker q(1-p)$.

Proof. By the previous lemma,

$$(1-p)\tilde{\pi}(a)(1-p) = \tilde{\pi}(a)(1-p) \in \ker q.$$

Thus, $\rho(a) = \tilde{\pi}(a)(1-p)$ defines a homomorphism $\rho : C(\Gamma) \to (1-p)\ker q(1-p) \subset \ker q$. Using the previous lemma,

$$\begin{aligned}(1-p)\ker q(1-p) &= \overline{\operatorname{span}}\{(1-p)T^{\otimes k}(\xi)(1-p)T^{\otimes k'}(\eta)^*(1-p) \,\big|\, k, k' \geq 0, \xi \in \mathcal{E}^{\otimes k}, \eta \in \mathcal{E}^{\otimes k'}\} \\ &= \overline{\operatorname{span}}\{(1-p)\tilde{\pi}(a)(1-p)\tilde{\pi}(b)^*(1-p) \,\big|\, a, b \in C(\Gamma)\} \\ &= \overline{\operatorname{span}}\{\tilde{\pi}(a)(1-p) \,\big|\, a \in C(\Gamma)\} = \operatorname{ran}\, \rho.\end{aligned}$$

Hence, ρ is surjective.

In order to show the injectivity of ρ, choose a faithful representation $\pi_0 : C(\Gamma) \to B(H)$ and consider the Fock representation (T_F, π_F) of $\mathcal{E}_{\alpha,\beta}(\Gamma)$ induced from π_0 as described in [27, Example 1.4]. The underlying space of this Fock representation is $F(\mathcal{E}_{\alpha,\beta}(\Gamma)) \otimes_A H := \oplus_{k\geq 0}(\mathcal{E}^{\otimes k} \otimes_A H)$ where $A = C(\Gamma)$ acts diagonally on the left and $\mathcal{E}_{\alpha,\beta}(\Gamma)$ acts by creation operators. Then $T_F(\xi)^*$ is an annihilation operator vanishing on the subspace $A \otimes_A H$ of $F(\mathcal{E}_{\alpha,\beta}(\Gamma)) \otimes_A H$. Now, for $a \in A$,

$$0 = (T_F \times \pi_F)(\rho(a)) = (T_F \times \pi_F)(\tilde{\pi}(a)(1-p) = \pi_F(a)(1 - \sum_{j=0}^{N-1} T_F(u_j)T_F(u_j)^*).$$

Since $T_F(u_j)^*$ vanishes on $A \otimes_A H$, we have that $\rho(a) = 0$ implies

$$\pi_F(a)(1 - \sum_{j=0}^{N-1} T_F(u_j)T_F(u_j)^*)(1 \otimes_A h) = 0$$

for all $h \in H$ and so, $\pi_F(a)(1 \otimes_A h) = 0$ for all $h \in H$. Thus, $a \otimes_A h = 0$ for all $h \in H$ and hence, $\pi_0(a)h = 0$ for all $h \in H$ which implies $a = 0$ since π_0 is faithful. Hence, ρ is injective.

□

Lemma 3.4. [24, Lemma 3.5] Suppose that A is a C^*-algebra, $r \geq 1$ and $N \geq 2$ are integers, and

$$\{b_{j,s;k,t} \,\big|\, 0 \leq j, k < N \text{ and } 0 \leq s, t < r\}$$

is a subset of A. For m, n satisfying $0 \leq m, n < rN - 1$, define

$$c_{m,n} = b_{j,s;k,t} \text{ where } m = sN + j \text{ and } n = lN + k, \text{ and}$$

$$d_{m,n} = b_{j,s;k,t} \text{ where } m = jr + s \text{ and } n = kr + t.$$

Then there is a scalar unitary permutation matrix U such that the matrices $C := (c_{m,n})_{m,n}$ and $D_{m,n} := (d_{m,n})_{m,n}$ are related by $C = UDU^*$.

The following is standard (and also appears in [24]):

Lemma 3.5. Suppose that S is an isometry in a unital C^*-algebra A. Then

$$U := \begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix}$$

is a unitary element of $M_2(A)$ and its class in $K_1(A)$ is the identity.

Proposition 3.6. Let $(T, \tilde{\pi})$ denote the universal Toeplitz representation on $\mathcal{E}_{\alpha,\beta}(\Gamma)$ and let $\{u_j\}_{j=0}^{N-1}$ be an orthonormal basis of $\mathcal{E}_{\alpha,\beta}(\Gamma)$. Further, let $p = \sum_{j=0}^{N-1} T_j T_j^*$ where $T_j = T(u_j)$. Then, with the maps $\Omega : C(\Gamma) \to M_N(C(\Gamma))$ and $\rho : C(\Gamma) \to \ker q \subset \mathcal{T}_{\alpha,\beta}(\Gamma)$ defined by

$$\Omega(a) = (\langle u_i, a \cdot u_j \rangle)_{i,j=0}^{N-1}$$

and

$$\rho(a) = \tilde{\pi}(a)(1 - p)$$

as in Lemmas 3.1 and 3.3, the following two diagrams ($i = 0, 1$) commute:

$$\begin{array}{ccc} K_i(C(\Gamma)) & \xrightarrow{1-\Omega_*} & K_i(C(\Gamma)) \\ \downarrow \rho_* & & \downarrow \tilde{\pi}_* \\ K_i(\ker q) & \xrightarrow{\iota_*} & K_i(\mathcal{T}_{\alpha,\beta}(\Gamma)) \end{array} \tag{3.3}$$

Proof. First, let $i = 0$. Let $z = (z_{s,t}) \in M_r(C(\Gamma))$ be a projection and let $\tilde{\pi}_r$ denote the augmentation map, $\tilde{\pi} \otimes \mathrm{id}_r$, of $\tilde{\pi}$ on $M_r(C(\Gamma))$. Then

$$\rho_*([z]) = [(\rho(z_{s,t}))_{s,t}] = [(\tilde{\pi}(z_{s,t})(1-p))_{s,t}] = [\tilde{\pi}_r(z)] - [\tilde{\pi}_r(z)(p1_r)]$$

and

$$\tilde{\pi}_* \circ (1 - \Omega_*)([z]) = [\tilde{\pi}_r(z)] - \tilde{\pi}_* \circ \Omega_*([z]).$$

Hence, it suffices to show that

$$\tilde{\pi}_* \circ \Omega_*([z]) = [\tilde{\pi}_r(z)(p1_r)].$$

Note that

$$\Omega_*([z]) = [(\Omega(z_{s,t}))_{s,t}] = [((\langle u_j, z_{s,t} \cdot u_k \rangle)_{j,k})_{s,t}],$$

so

$$\tilde{\pi}_* \circ \Omega_*([z]) = [((\tilde{\pi}(\langle u_j, z_{s,t} \cdot u_k \rangle))_{j,k})_{s,t}] = [\tilde{\pi}_{rN} \circ \Omega_r(z)].$$

Set $b_{j,s;k,t} = \tilde{\pi}(\langle u_j, z_{s,t} \cdot u_k \rangle)$ and $C = (c_{m,n})_{m,n} = \tilde{\pi}_{rN}(\Omega_r(z))$ as in Lemma 3.4.

Let

$$T = \begin{pmatrix} T_0 1_r & T_1 1_r & \dots & T_{N-1} 1_r \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \in M_N(M_r(\mathcal{O}_{\alpha,\beta}(\Gamma))).$$

Then $TT^* = p1_r \oplus 0_{r(N_1)}$ and since $\tilde{\pi}_r(z)$ is a projection which commutes with $p1_r$,

$$(\tilde{\pi}_r(z) \oplus 0_{r(N-1)})T$$

is a partial isometry which implements a Murray-von Neumann equivalence between

$$T^*(\tilde{\pi}_r(z) \oplus 0_{r(N-1)})T$$

and

$$(\tilde{\pi}_r(z) \oplus 0_{r(N-1)})TT^*(\tilde{\pi}_r(z) \oplus 0_{r(N-1)}) = \tilde{\pi}_r(z)(p1_r) \oplus 0_{r(N-1)};$$

thus,

$$[\tilde{\pi}_r(z)(p1_r)] = [T^*(\tilde{\pi}_r(z) \oplus 0_{r(N-1)})T].$$

Furthermore,

$$T^*(\tilde{\pi}_r(z) \oplus 0_{r(N-1)})T = T^* \begin{pmatrix} \tilde{\pi}_r(z)T_0 & \dots & \tilde{\pi}_r(z)T_{N-1} \\ 0 & \dots & 0 \\ \vdots & \dots & \vdots \end{pmatrix} = (T_j^* \tilde{\pi}_r(z) T_k)_{j,k}$$

so the (j,k) entry is $(\tilde{\pi}(\langle u_j, z_{s,t} \cdot u_k \rangle))_{s,t}$. Recall $b_{j,s;k,t} = \tilde{\pi}(\langle u_j, z_{s,t} \cdot u_k \rangle)$ and so $T^*(\tilde{\pi}_r(z) \oplus 0_{r(N-1)})T = D = (d_{m,n})_{m,n}$ as in Lemma 3.4. Thus, by Lemma 3.4, there exists a unitary U such that $C = U^*DU$ which gives us

$$[\tilde{\pi}_r(z)(p1_r)] = [D] = [C] = [\tilde{\pi}_{rN} \circ \Omega_R(z)]$$

as desired.

For the case $i = 1$, let $u \in M_r(C(\Gamma))$ be a unitary. Note $\rho_* : K_1(C(\Gamma)) \to K_1(\ker q)$ is the composition of a unital isomorphism of $C(\Gamma)$ onto $(1-p)\ker q(1-p)$ with the inclusion of $(1-p)\ker q(1-p)$ as a full corner in the non-unital algebra $\ker q$; that is, $[u] \mapsto [\rho_r(u)] = [\tilde{\pi}_r(u)((1-p)1_r)] \mapsto [\tilde{\pi}_r(u)((1-p)1_r)+p1_r] \in K_1((\ker q)^+) = K_1(\ker q)$. Furthermore,

$$\tilde{\pi}_* \circ \Omega_*([u]) = [\tilde{\pi}_r(u)] - [\tilde{\pi}_{rN} \circ \Omega_r(u)]$$

and hence, we need only show

$$[(\tilde{\pi}_r(u)((1-p)1_r) + p1_r) \oplus 1_{r(N-1)}] = [\tilde{\pi}_r(u) \oplus 1_{r(N-1)}] - [\tilde{\pi}_{rN} \circ \Omega_r(u)]$$

in $K_1(\mathcal{T}_{\alpha,\beta}(G))$.

We take a brief moment to make an aside: If $C \in M_{2rN}(\mathcal{T}_{\alpha,\beta}(\Gamma))$ is invertible with K_1-class the identity 1 then the K_1-class is unchanged by pre- and post-multiplication by C. In particular, when C is equal to:

(1) (Lemma 3.5) a unitary of the form

$$\begin{pmatrix} S & 1 - SS^* \\ 0 & S^* \end{pmatrix}$$

where S is an isometry

(2) an upper- or lower-triangular matrix of the form

$$\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 \\ A & 1 \end{pmatrix}$$

(which are connected to 1 via $t \mapsto \begin{pmatrix} 1 & tA \\ 0 & 1 \end{pmatrix}$ and likewise for the transpose)

(3) any constant invertible matrix in $\mathrm{GL}_{2rN}(\mathbb{C})$ (because $\mathrm{GL}_{2rN}(\mathbb{C})$ is path connected); this implies that row and column operations may be used without changing the K_1-class.

$$\text{Recall: } T = \begin{pmatrix} T_0 1_r & T_1 1_r & \dots & T_{N-1} 1_r \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

With this in mind, calculate

$$\begin{aligned}
&[(\tilde{\pi}_r(u)((1-p)1_r)+p1_r)\oplus 1_{r(N-1)}] \\
&\quad = \left[\begin{pmatrix} (\tilde{\pi}_r(u)((1-p)1_r)+p1_r)\oplus 1_{r(N-1)} & 0_{rN} \\ 0_{rN} & 1_{rN}\end{pmatrix}\right]\left[\begin{pmatrix} T & 1_{rN}-TT^* \\ 0_{rN} & T^*\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} (\tilde{\pi}_r(u)((1-p)1_r)+p1_r)\oplus 1_{r(N-1)} & 0_{rN} \\ 0_{rN} & 1_{rN}\end{pmatrix}\right]\left[\begin{pmatrix} T & (1-p)1_r\oplus 1_{r(N-1)} \\ 0_{rN} & T^*\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} ((\tilde{\pi}_r(u)(1-p)1_r)+p1_r)\oplus 1_{r(N-1)})T & \tilde{\pi}(u)((1-p)1_r)\oplus 1_{r(N-1)} \\ 0_{rN} & T^*\end{pmatrix}\right]
\end{aligned}$$

and recall $(1-p)T_i = 0$ by Lemma 3.2(3), hence $(1-p)1_rT = 0$ and

$$\begin{aligned}
&[(\tilde{\pi}_r(u)((1-p)1_r)+p1_r)\oplus 1_{r(N-1)}] \\
&\quad = \left[\begin{pmatrix} T & \tilde{\pi}_r(u)((1-p)1_r)\oplus 1_{r(N-1)} \\ 0_{rN} & T^*\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} T & \tilde{\pi}_r(u)((1-p)1_r)\oplus 1_{r(N-1)} \\ 0_{rN} & T^*\end{pmatrix}\right]\left[\begin{pmatrix} 1_{rN} & T^*(\tilde{\pi}_r(u)\oplus 1_{r(N-1)}) \\ 0_{rN} & 1_{rN}\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} T & \tilde{\pi}_r(u)\oplus 1_{r(N-1)} \\ 0_{rN} & T^*\end{pmatrix}\right]
\end{aligned}$$

since $TT^* = p1_r \oplus 0_{r(N-1)}$ and $(p1_r)\tilde{\pi}_r(u) = \tilde{\pi}_r(u)(p1_r)$. Using elementary operations, compute

$$\begin{aligned}
&[(\tilde{\pi}_r(u)((1-p)1_r)+p1_r)\oplus 1_{r(N-1)}] \\
&\quad = \left[\begin{pmatrix} \tilde{\pi}_r(u)\oplus 1_{r(N-1)} & T \\ T^* & 0_{rN}\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} \tilde{\pi}_r(u)\oplus 1_{r(N-1)} & T \\ T^* & 0_{rN}\end{pmatrix}\right]\left[\begin{pmatrix} 1_{rN} & -(\tilde{\pi}_r(u*)\oplus 1_{r(N-1)})T \\ 0_{rN} & 1_{rN}\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} \tilde{\pi}_r(u)\oplus 1_{r(N-1)} & 0_{rN} \\ T^* & -T^*(\tilde{\pi}_r(u^*)\oplus 1_{r(N-1)})T\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} 1_{rN} & 0_{rN} \\ -T^*(\tilde{\pi}_r(u^*)\oplus 1_{r(N-1)}) & 1_{rN}\end{pmatrix}\right]\left[\begin{pmatrix} \tilde{\pi}_r(u)\oplus 1_{r(N-1)} & 0_{rN} \\ T^* & -T^*(\tilde{\pi}_r(u^*)\oplus 1_{r(N-1)})T\end{pmatrix}\right] \\
&\quad = \left[\begin{pmatrix} \tilde{\pi}_r(u)\oplus 1_{r(N-1)} & 0_{rN} \\ 0_{rN} & -T^*(\tilde{\pi}_r(u^*)\oplus 1_{r(N-1)})T\end{pmatrix}\right]\left[\begin{pmatrix} 1_{rN} & 0_{rN} \\ 0_{rN} & -1_{rN}\end{pmatrix}\right] \\
&\quad = [\tilde{\pi}_r(u)\oplus 1_{r(N-1)}] + [T^*(\tilde{\pi}_r(u^*)\oplus 1_{r(N-1)})T].
\end{aligned}$$

Furthermore,

$$[T^*(\tilde{\pi}_r(u^*)\oplus 1_{r(N-1)})T] = [\tilde{\pi}_{rN}(\Omega_r(u^{-1}))] = -[\tilde{\pi}_{rN}(\Omega_r(u))].$$

Hence,

$$[(\tilde{\pi}_r(u)((1-p)1_r)+p1_r)\oplus 1_{r(N-1)}] = [\tilde{\pi}_r(u)\oplus 1_{r(N-1)}] - [\tilde{\pi}_{rN}\circ\Omega_r(u)]$$

as desired.

□

Theorem 3.7. Let $(S,\pi) = q\circ(T,\tilde{\pi})$ be the universal Cuntz-Pimsner covariant representation of $\mathcal{E}_{\alpha,\beta}(\Gamma)$ in $\mathcal{O}_{\alpha,\beta}(\Gamma)$. Then the following diagram is exact:

$$\begin{array}{ccccc} K_0(C(\Gamma)) & \xrightarrow{1-\Omega_*} & K_0(C(\Gamma)) & \xrightarrow{\pi_*} & K_0(\mathcal{O}_{\alpha,\beta}(\Gamma)) \\ {\scriptstyle \rho_*^{-1}\circ\delta_0}\uparrow & & & & \downarrow{\scriptstyle \rho_*^{-1}\circ\delta_1} \\ K_1(\mathcal{O}_{\alpha,\beta}(\Gamma)) & \xleftarrow[\pi_*]{} & K_1(C(\Gamma)) & \xleftarrow[1-\Omega_*]{} & K_1(C(\Gamma)) \end{array} \tag{3.4}$$

Proof. Note $\rho: C(\Gamma)\to \ker q$ is an isomorphism onto a full corner, implying ρ_* is an isomorphism. Further note $\tilde{\pi}_*: K_i(C(\Gamma))\to K_i(\mathcal{T}_{\alpha,\beta}(\Gamma))$ is an isomorphism (see comments prior to Lemma 3.1). Then (3.1) and the previous proposition give the stated result.

□

4. K-groups of $\mathcal{O}_{F,G}(\Gamma)$

In this section, the approach of [24] is made easier and extended. For this section, let $\alpha,\beta\in \mathrm{End}(C(\mathbb{T}^d))$ defined by $\alpha=\sigma_F^{\#}$ and $\beta=\sigma_G^{\#}$ where $F=\mathrm{Diag}(a_1,...,a_d)\in M_d(\mathbb{Z})^+$ and $G\in M_d(\mathbb{Z})$ such that $\det F>0$ and $\det G\neq 0$. We know there exists an orthonormal basis for $\mathcal{E}_{F,G}(\mathbb{T}^d)$, $\{u_j \,\big|\, 0\le j\le N-1\}$; this is the basis $\{u_\nu\}_{\nu\in\mathfrak{I}(F)}$, described in Section 3.3, reindexed by $0,1,...,N-1$. Let U_j be the unitary defined by $U_j(x)=x_j$ for $x=(x_i)_{i=1}^d\in\mathbb{T}^d$ for $j\in\{1,...,d\}$. Further, let

$$\mathfrak{I}_k=\{J\subset\{1,...,d\}\,\big|\,|J|=k, J=\{j_1<...<j_k\}\}$$

and $J'=\{1,...,d\}\setminus J$ in increasing order. Define

$$\mathfrak{E}_k=\begin{cases}\{[1]_0\} & \text{if } k=0\\ \{[U_J]_0=[U_{j_1}]_0\wedge...\wedge[U_{j_k}]_0 \,\big|\, J\in\mathfrak{I}_k\} & \text{if } k>0 \text{ is even}\\ \{[U_J]_1=[U_{j_1}]_1\wedge...\wedge[U_{j_k}]_1 \,\big|\, J\in\mathfrak{I}_k\} & \text{if } k>0 \text{ is odd}\end{cases}$$

If it is understood, the notation $[\cdot]$ will be used in lieu of $[\cdot]_i$.

It is well known (see [34] and [33, Example 3.11 and 3.15]) that

$$K_0(C(\mathbb{T}^d))\cong\bigwedge_{\text{evens}}\mathbb{Z}^d=\mathbb{Z}^{2^{d-1}}$$

with basis $\{\mathfrak{E}_k\}_{k \text{ even}}$ and

$$K_1(C(\mathbb{T}^d))\cong\bigwedge_{\text{odds}}\mathbb{Z}^d=\mathbb{Z}^{2^{d-1}}$$

with basis $\{\mathfrak{E}_k\}_{k \text{ odd}}$. For subsets J and I of the same size, define $F_{J,I}$ to be the square submatrix of F whose entries belong to the rows in J and the columns in I.

With these identifications, the (K_1-group) induced map $\alpha_*|_{\bigwedge^1\mathbb{Z}^d}:\mathrm{span}\{[U_j]\}_{j=1}^d\to\mathrm{span}\{[U_j]\}_{j=1}^d$ is multiplication by $F^T=F$, and $\beta_*|_{\bigwedge^1\mathbb{Z}^d}:\mathrm{span}\{[U_j]\}_{j=1}^d\to\mathrm{span}\{[U_j]\}_{j=1}^d$ is multiplication by G^T, the transpose of G. We have

$$\beta_*([U_j])=[\beta(U_j)]=[U^{G_j}]=[\prod_k U_k^{b_{jk}}]=\sum_k b_{jk}[U_k].$$

Do likewise to prove $\alpha_*|_{\bigwedge^1 \mathbb{Z}^d}$ is multiplication by F. One can also check that α_* and β_* act on $\bigwedge^0 \mathbb{Z}^d$ by

$$\alpha_*[1] = [\alpha(1)] = 1 = [\beta(1)] = \beta_*[1]$$

since α and β are group homomorphisms.

Lemma 4.1. For $1 \leq k \leq d$, the matrix A_k representating $\alpha_*| : \bigwedge^k \mathbb{Z}^d \to \bigwedge^k \mathbb{Z}^d$ with respect to the basis $\mathfrak{E}_k$ is the diagonal matrix $\mathrm{Diag}(a_I)_{I \in \mathfrak{I}_k}$ ($a_I = \prod_{i \in I} a_i$) and matrix B_k representating $\beta_*| : \bigwedge^k \mathbb{Z}^d \to \bigwedge^k \mathbb{Z}^d$ is $(\det G_{JI})_{I,J \in \mathfrak{I}_k}$.

Proof. Begin by noting, $A_k = \bigwedge^k F^T$ and $B_k = \bigwedge^k G^T$. Let $[U_I] \in \mathfrak{E}_k$ with $I = \{i_1 < ... < i_k\}$. Then

$$\begin{aligned}
\beta_*([U_I]) &= (\bigwedge^k G^T)[U_I] = (\bigwedge^k G^T)([U_{i_1}] \wedge ... \wedge [U_{i_k}]) \\
&= G^T[U_{i_1}] \wedge ... \wedge G^T[U_{i_k}] \\
&= \sum_{m_1,...,m_k=1}^{d} b_{i_1,m_1} ... b_{i_k,m_k} ([U_{m_1}] \wedge ... \wedge [U_{m_k}]) \\
&= \sum_{[U_J] \in \mathfrak{E}_k, J=\{m_1,...,m_k\}} b_{i_1,m_1} ... b_{i_k,m_k} [U_J] \\
&= \sum_{[U_J] \in \mathfrak{E}_k} \sum_{\sigma \in S_k} b_{i_1,\sigma(j_1)} ... b_{i_k,\sigma(j_k)} ([U_{\sigma(j_1)}] \wedge ... \wedge [U_{\sigma(j_k)}]) \\
&= \sum_{[U_J] \in \mathfrak{E}_k} \sum_{\sigma \in S_k} (-1)^{\deg \sigma} b_{i_1,\sigma(j_1)} ... b_{i_k,\sigma(j_k)} ([U_{j_1}] \wedge ... \wedge [U_{j_k}]) \\
&= \sum_{[U_J] \in \mathfrak{E}_k} \det G_{I,J} [U_J]
\end{aligned}$$

where S_k denotes the symmetric group on k elements. The result for A_k follows by specializing to the diagonal case.

□

Let A_k and B_k (for $k = 0, ..., d$) denote the matrices described in Lemma 4.1; that is, $A_0 = B_0 = 1$, $A_k = \mathrm{Diag}(a_I)_{I \in \mathfrak{I}_k}$ and $B_k = (\det G_{JI})_{I,J \in \mathfrak{I}_k}$ for $k \in \{1, ..., d\}$. From Lemma 3.1, the map $\Omega(\sigma_F^{\#}) : C(\mathbb{T}^d) \to M_N(C(\mathbb{T}^d))$ is the diagonal matrix $\Omega(\sigma_F^{\#}(a)) = \sigma_G^{\#}(a) 1_N$ and so, for $\alpha = \sigma_F^{\#}$, $\beta = \sigma_G^{\#}$ and $d = (d_{s,t}) \in M_r(C(\mathbb{T}^d))$,

$$\begin{aligned}
\Omega_* \circ \alpha_*([d]) &= [(\Omega \circ \alpha)_r(d)] \\
&= [(\Omega(\alpha(d_{s,t}))_{s,t}] \\
&= [(\beta(d_{s,t}) 1_N)_{s,t}] \\
&= N[\beta_r(d)] \\
&= N\beta_*[d]
\end{aligned}$$

where $\Omega_r, \alpha_r, \beta_r$ denote the appropriate augmented maps on $M_r(C(\mathbb{T}^d))$. Using the previous lemma and the above equation, the matrix C_k representing Ω_* on $\bigwedge^k \mathbb{Z}^d$

satisfies

$$C_k A_k = N B_k$$

where it is expected that C_k is a matrix with integer entries for each $k = 0, ..., d$.

Recall $A_k = \mathrm{Diag}(a_I)_{I\in\mathfrak{I}_k}$. Let $I' = \{1, ..., d\} \setminus I$ ordered so that $I = \{i_1 < ... < i_k\}, I' = \{i_{k+1} < ... < i_d\}$. Then set $C_0 = N = \det F \in \mathbb{Z}$ and

$$C_k = B_k \mathrm{Diag}(a_{I'})_{I\in\mathfrak{I}_k} = (a_{J'} \det G_{JI})_{I,J\in\mathfrak{I}_k}$$

for $k = 1, ..., d$. Note that $A_0 = B_0 = 1$ by the calculations before Lemma 4.1. Hence,

$$C_0 A_0 = N B_0$$

and

$$\begin{aligned} C_k A_k &= B_k \mathrm{Diag}(a_{I'})_{I\in\mathfrak{I}_k} \mathrm{Diag}(a_I)_{I\in\mathfrak{I}_k} \\ &= B_k \mathrm{Diag}(a_{I\cup I'})_{I\in\mathfrak{I}_k} \\ &= B_k \mathrm{Diag}(a_{1,...,d})_{I\in\mathfrak{I}_k} \\ &= B_k \mathrm{Diag}(N)_{I\in\mathfrak{I}_k} = N B_k \end{aligned}$$

for $k = 1, ..., d$.

In order to calculate the K-theory of $\mathcal{O}_{F,G}(\mathbb{T}^d)$, one needs only to calculate $\ker(1-C_k)$ and $\mathrm{coker}(1 - C_k)$ for each $k = 0, ..., d$ as the next theorem will demonstrate.

Theorem 4.2. Let $F = \mathrm{Diag}(a_1, ..., a_d), G \in M_d(\mathbb{Z})$ such that $\det G \neq 0$ and $a_j \in \mathbb{N}$ for each $j = 1, ..., d$. With C_k defined as above and $\mathcal{O}_{F,G}(\mathbb{T}^d)$ defined as in Example 2.17,

(1) $K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \Big(\bigoplus_{0\le k\le d,\,\mathrm{even}} \mathrm{coker}(1 - C_k)\Big) \oplus \Big(\bigoplus_{0\le k\le d,\,\mathrm{odd}} \ker(1 - C_k)\Big)$, and

(2) $K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \Big(\bigoplus_{0\le k\le d,\,\mathrm{odd}} \mathrm{coker}(1 - C_k)\Big) \oplus \Big(\bigoplus_{0\le k\le d,\,\mathrm{even}} \ker(1 - C_k)\Big)$.

Proof. By 3.4 in Theorem 3.7,

$$\begin{tikzcd} K_0(C(\mathbb{T}^d) \arrow[r, "\bigoplus_{0\le k\le d,\,\mathrm{even}} 1 - C_k"] & K_0(C(\mathbb{T}^d)) \arrow[r] & K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) \arrow[d] \\ K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) \arrow[u] & K_1(C(\mathbb{T}^d)) \arrow[l] & K_1(C(\mathbb{T}^d)) \arrow[l, "\bigoplus_{0\le k\le d,\,\mathrm{odd}} 1 - C_k"] \end{tikzcd}$$

is exact, so there are two exact sequences

$$0 \longrightarrow \bigoplus_{0\le k\le d,\,\mathrm{even}} \mathrm{coker}(1 - C_k) \longrightarrow K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) \longrightarrow \bigoplus_{0\le k\le d,\,\mathrm{odd}} \ker(1 - C_k) \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{0\le k\le d,\,\mathrm{odd}} \mathrm{coker}(1 - C_k) \longrightarrow K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) \longrightarrow \bigoplus_{0\le k\le d,\,\mathrm{even}} \ker(1 - C_k) \longrightarrow 0$$

which split since $\bigwedge^k \mathbb{Z}^d$ and hence, $\ker(1-C_k)$ is free for each k. Thus,

$$K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \big(\bigoplus_{0\le k\le d,\,\text{even}} \operatorname{coker}(1-C_k)\big) \oplus \big(\bigoplus_{0\le k\le d,\,\text{odd}} \ker(1-C_k)\big)$$

and

$$K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \big(\bigoplus_{0\le k\le d,\,\text{odd}} \operatorname{coker}(1-C_k)\big) \oplus \big(\bigoplus_{0\le k\le d,\,\text{even}} \ker(1-C_k)\big).$$

Definition 4.3. A matrix $Z \in M_d(\mathbb{Z})$ is called an *integer dilation matrix* provided each eigenvalue λ of Z satisfies $|\lambda| > 1$.

Remark 4.4. The case where F is an integer dilation and $G = 1_d$ was computed in [24, Theorem 4.9] where it was found that

$$K_0(\mathcal{O}_{F,1}(\mathbb{T}^d)) = \big(\bigoplus_{0\le k\le d,\,\text{even}} \operatorname{coker}(1-Q_k)\big) \oplus \big(\bigoplus_{0\le k\le d,\,\text{odd}} \ker(1-Q_k)\big)$$

and

$$K_1(\mathcal{O}_{F,1}(\mathbb{T}^d)) = \big(\bigoplus_{0\le k\le d,\,\text{odd}} \operatorname{coker}(1-Q_k)\big) \oplus \big(\bigoplus_{0\le k\le d,\,\text{even}} \ker(1-Q_k)\big)$$

for the matrix Q_k satisfying the relation

$$Q_k(\det F_{JI})_{I,J\in\mathfrak{I}_k} = N1_{\binom{d}{k}}.$$

Recall the Smith normal form of F, $F = UDV$ where $U, V \in M_d(\mathbb{Z})$ are unimodular matrices and $D = \operatorname{Diag}(a_j)_{j=1}^d \in M_d(\mathbb{Z})$ is a positive diagonal matrix. Then by properties of matrix minors,

$$(\det F_{JI})_{I,J\in\mathfrak{I}_k} = U_k D_k V_k$$

where $U_k = (\det U_{JI})_{I,J\in\mathfrak{I}_k}$, $V_k = (\det V_{JI})_{I,J\in\mathfrak{I}_k}$ and $D_k = \operatorname{Diag}(a_I)_{I\in\mathfrak{I}_k}$. Also note for $(U^{-1})_k = (\det(U^{-1})_{JI})_{I,J\in\mathfrak{I}_k}$,

$$U_k(U^{-1})_k = (UU^{-1})_k = 1_{\binom{d}{k}};$$

that is, U_k is unimodular. Hence, for $G = U^{-1}V^{-1}$, $Q_kU_kD_kV_k = N1_{\binom{d}{k}}$ implies

$$U_k^{-1}Q_kU_k = U_k^{-1}V_k^{-1}\operatorname{Diag}(a_{I'})_{I\in\mathfrak{I}_k} = B_k\operatorname{Diag}(a_{I'})_{I\in\mathfrak{I}_k} = C_k.$$

Therefore,

$$\ker(1-C_k) = \ker(U_k^{-1}(1-Q_k)U_k) \cong \ker(1-Q_k)$$

and likewise,

$$\operatorname{coker}(1-C_k) \cong \operatorname{coker}(1-Q_k);$$

that is, Theorem 4.2 extends Theorem 4.9 of [24].

Here we first consider the case where $F = n1_d$ and $G \in M_d(\mathbb{Z})$. To this end, note that C_k has a simple form. First of all, note A_k is $n^k 1_{\binom{d}{k}}$ and hence,

$$C_k = n^{d-k}B_k. \qquad \text{(see the calculations preceding Theorem 4.2.)}$$

Theorem 4.5. For $d \in \mathbb{N}$, let $G \in M_d(\mathbb{Z})$ $(\det G \neq 0)$ and $n \in \mathbb{N}$. Then with $C_k = n^{d-k} B_k$ where $B_k = (\det G_{J,I})_{I,J \in \mathfrak{I}_k} \in M_{\binom{d}{k}}(\mathbb{Z})$ as above,

(1) If $d > 1$ and $n > 1$, then

(a) $K_0(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \begin{cases} \bigoplus_{0 \leq k \leq d, \text{ even}} \operatorname{coker}(1 - C_k) & \text{if } \det G \neq 1 \\ \mathbb{Z} \oplus \big(\bigoplus_{0 \leq k \leq d-1, \text{ even}} \operatorname{coker}(1 - C_k) \big) & \text{if } \det G = 1 \end{cases}$

(b) $K_1(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \begin{cases} \bigoplus_{0 \leq k \leq d, \text{ odd}} \operatorname{coker}(1 - C_k) & \text{if } \det G \neq 1 \\ \mathbb{Z} \oplus \big(\bigoplus_{0 \leq k \leq d-1, \text{ odd}} \operatorname{coker}(1 - C_k) \big) & \text{if } \det G = 1 \end{cases}$

(2) If $n = 1$ and G is an integer dilation matrix, then

(a) $K_0(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \mathbb{Z} \oplus \big(\bigoplus_{1 \leq k \leq d, \text{ even}} \operatorname{coker}(1 - C_k) \big)$

(b) $K_1(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \mathbb{Z} \oplus \big(\bigoplus_{1 \leq k \leq d, \text{ odd}} \operatorname{coker}(1 - C_k) \big)$

(3) If $d = 1$, then

(a) $K_0(\mathcal{O}_{n,m}(\mathbb{T})) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_{n,m}(\mathbb{T})) = \mathbb{Z}_{m-1}$ for $n > 1$ and $m \neq 0, 1$

(b) $K_0(\mathcal{O}_{1,m}(\mathbb{T})) = \mathbb{Z}$ and $K_1(\mathcal{O}_{1,m}(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}_{m-1}$ for $m \neq 0, 1$

(c) $K_0(\mathcal{O}_{n,1}(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_{n,1}(\mathbb{T})) = \mathbb{Z}$ for $n > 1$

Proof. For (1), let $d, n > 1$ and $\det G \neq 1$. Then $C_0 = n^d \neq 1$ and $C_d = \det G \neq 1$; that is, $1 - C_0$ and $1 - C_d$ are injective. Furthermore, for $1 \leq k \leq d-1$,

$$C_k = n^{d-k} (\det G_{J,I})_{J,I \in \mathfrak{I}_k}.$$

Since the characteristic polynomial of a matrix is monic, it follows from Gauss' Lemma that any rational eigenvalue of a matrix in $M_d(\mathbb{Z})$ must actually be an integer. That is, $\frac{1}{n^{d-k}} \notin \sigma((\det G_{J,I})_{I,J})$ if and only if $1 \notin \sigma(C_k)$. Thus, $1 - C_k$ is injective for each $k = 1, ..., d-1$ and so, $\ker(1 - C_k) = 0$ for each $k = 0, ..., d$. By Theorem 4.2,

$$K_0(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \bigoplus_{0 \leq k \leq d, \text{ even}} \operatorname{coker}(1 - C_k),$$

and

$$K_1(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \bigoplus_{0 \leq k \leq d, \text{ odd}} \operatorname{coker}(1 - C_k).$$

Now assume $\det G = 1$, then, as above, $\ker(1 - C_k) = 0$ for $k = 0, ..., d-1$. But $\ker(1 - C_d) = \mathbb{Z}$ and $\operatorname{coker}(1 - C_d) = \mathbb{Z}$ whether d is even or odd. Theorem 4.2 gives

$$K_0(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \mathbb{Z} \oplus \big(\bigoplus_{0 \leq k \leq d-1, \text{ even}} \operatorname{coker}(1 - C_k) \big)$$

and

$$K_1(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \mathbb{Z} \oplus \big(\bigoplus_{0 \leq k \leq d-1, \text{ odd}} \operatorname{coker}(1 - C_k) \big).$$

For (2), let $n = 1$ and $|\lambda| > 1$ for all eigenvalues λ of G. Then $C_0 = 1$ and $C_d = \det G \neq 1$. We now wish to show that $\det(1 - C_k) \neq 0$ for $k = 1, ..., d$. To this end, choose a basis of $\mathbb{C}^d$ such that G becomes upper triangular (not necessarily with integer entries); that is,

$$G = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1d} \\ 0 & a_{22} & \dots & a_{2d} \\ \vdots & \ddots & \ddots & \\ 0 & 0 & \dots & a_{dd} \end{pmatrix}.$$

Then C_k is lower triangular with diagonal entries $\det G_{II} = \prod_{i\in I} a_{ii} \neq 1$ (since $|\lambda| > 1$ for all $\lambda \in \sigma(G)$) and so $\det(1 - C_k) \neq 0$. Hence, $1 - C_k$ is injective for $k = 1, ..., d$ and $\ker(1 - C_0) = \operatorname{coker}(1 - C_0) = \mathbb{Z}$. Theorem 4.2 now gives

$$K_0(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \mathbb{Z} \oplus \big(\bigoplus_{1\leq k\leq d,\text{ even}} \operatorname{coker}(1 - C_k)\big),$$

and

$$K_1(\mathcal{O}_{n,G}(\mathbb{T}^d)) = \mathbb{Z} \oplus \big(\bigoplus_{1\leq k\leq d,\text{ odd}} \operatorname{coker}(1 - C_k)\big).$$

Finally, let us prove (3). If $n > 1$ and $m \neq 0, 1$ then $1 - n$ and $1 - m$ are injective and $\operatorname{coker}(1 - n) = \mathbb{Z}_{n-1} := \mathbb{Z}/(n - 1)\mathbb{Z}$; likewise, $\operatorname{coker}(1 - m) = \mathbb{Z}_{m-1}$. Thus, $K_0(\mathcal{O}_{n,m}(\mathbb{T})) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_{n,m}(\mathbb{T})) = \mathbb{Z}_{m-1}$. If $n = 1$ and $m \neq 0, 1$, then $\ker(1 - n) = \operatorname{coker}(1 - n) = \mathbb{Z}$, $\operatorname{coker}(1 - m) = \mathbb{Z}_{m-1}$ and $\ker(1 - m) = 0$ thus, $K_0(\mathcal{O}_{1,m}(\mathbb{T})) = \mathbb{Z}$ and $K_1(\mathcal{O}_{1,m}(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}_{m-1}$. If $n > 1$ and $m = 1$, then similarly, $K_0(\mathcal{O}_{n,1}(\mathbb{T})) = \mathbb{Z} \oplus \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_{n,1}(\mathbb{T})) = \mathbb{Z}$.

□

Corollary 4.6. Let $d = 2$, $n = 1$, and $1 \notin \sigma(G)$, the spectrum of G. Then

(1) if $\det G = 1$, then
  (a) $K_0(\mathcal{O}_{1,G}(\mathbb{T}^2)) = \mathbb{Z}^2$
  (b) $K_1(\mathcal{O}_{1,G}(\mathbb{T}^2)) = \mathbb{Z}^2 \oplus \operatorname{coker}(1 - G^T)$
(2) if $\det G \neq 1$, then
  (a) $K_0(\mathcal{O}_{1,G}(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}_{1-\det G}$
  (b) $K_1(\mathcal{O}_{1,G}(\mathbb{T}^2)) = \mathbb{Z} \oplus \operatorname{coker}(1 - G^T)$

Proof. Begin with calculating $C_0 = n^2 = 1$, $C_1 = nG^T = G^T$ and $C_2 = \det G$. With 1 not an eigenvalue of G, it is guaranteed that $\det(1 - G^T) \neq 0$ and hence, $1 - C_1$ is injective. This leaves us to calculate

(1) $\ker(1 - C_0) = \mathbb{Z}$
(2) $\operatorname{coker}(1 - C_0) = \mathbb{Z}$
(3) $\ker(1 - C_1) = 0$
(4) $\operatorname{coker}(1 - C_1) = \operatorname{coker}(1 - G^T)$
(5) $\ker(1 - C_2) = \begin{cases} \mathbb{Z} & \text{if } \det G = 1 \\ 0 & \text{if } \det G \neq 1 \end{cases}$, and

$$(6)\ \operatorname{coker}(1-C_2) = \begin{cases} \mathbb{Z} & \text{if } \det G = 1 \\ \mathbb{Z}_{1-\det G} & \text{if } \det G \neq 1 \end{cases}$$

□

We now calculate the K-theory when $F, G \in M_d(\mathbb{Z})$ are both diagonal matrices. Let $F = \operatorname{Diag}(a_1, ..., a_d)$ and $G = \operatorname{Diag}(b_1, ..., b_d)$ be diagonal integral matrices of non-zero determinant and such that $1 \leq a_1 \leq ... \leq a_d$. Let f denote the number of 1's in F; that is, $1 = a_1 = ... = a_f < a_{f+1} \leq ... \leq a_d$. Let $a_I = \prod_{i\in I} a_i$ for $I \in \mathfrak{I}_k$. Then $A_k = \operatorname{Diag}(a_I)_{I\in\mathfrak{I}_k}$, $B_k = \operatorname{Diag}(b_I)_{I\in\mathfrak{I}_k}$ and

$$C_k = (\det F)B_k A_k^{-1} = \operatorname{Diag}(b_I a_{I'})_{I\in\mathfrak{I}_k}.$$

So $\ker(1-C_k) = \mathbb{Z}^{d_k}$ where d_k is the number of 1's in C_k; that is, the number of I making $b_I a_{I'} = 1$. Furthermote, $b_I a_{I'} = 1$ implies $b_I = a_{I'} = 1$ and so $\{f+1, ..., d\} \subset I$. So let v be the number of negative ones in $\{b_{f+1}, ..., b_d\}$ and let p be the number of ones in the same set. Then

$$d_k = \sum_{r\in\mathfrak{I}} \binom{v}{2r}\binom{p}{(k-d+f)-2r}$$

where $\mathfrak{I} = \{r \in \mathbb{N} \,\big|\, 0 \leq 2r \leq v, p-(k-d+f) \leq 2r \leq k-d+f\}$.

Lemma 4.7. With the context of the preceding paragraph, let $p > 0$ be the number of ones while v is the number of negative ones. Let $p_k(p, v)$ be the number of combinations of choosing k digits of $1's$ and $-1's$ that multiply to 1 and let $v_k(p, v)$ be similar, but multiplying to -1; that is,

$$p_k(p, v) = \sum_{0, k-p \leq 2r \leq v, k} \binom{v}{2r}\binom{p}{k-2r}$$

and

$$v_k(p, v) = \sum_{0, k-p \leq 2r+1 \leq v, k} \binom{v}{2r+1}\binom{p}{k-2r-1}$$

for $0 < k \leq p$, $p_0(p, v) = 1$ if $p > 0$, and $v_0(p, v) = v_k(p, v) = p_k(p, v) = p_0(0, v) = 0$ for $k > p$. Then

$$\sum_{0\leq k\leq p} p_k(p, v) = \begin{cases} 2^p & \text{if } v = 0 \text{ and } p > 0 \\ 2^{p+v-1} & \text{if } v \neq 0 \end{cases}.$$

Proof. First realize that $p_k(p, v) + v_k(p, v) = \binom{p+v}{k}$ since it is the number of combinations of choosing k digits to multiply to either 1 or -1. Thus,

$$\sum_k p_k(p, v) + \sum_k v_k(p, v) = \sum_k \binom{p+v}{k} = 2^{p+v}.$$

So if $v = 0$, then the proof is done.

Assume $v \neq 0$ and let v be odd. Then claim $v_k(p, v) = p_{p-k}(p, v)$. This can be easily seen by realizing that, for each choice of k digits to multiply to -1, the

remaining digits multiply to 1. Thus, $\sum_k v_k(p,v) = \sum_k p_{p-k}(p,v) = \sum_k p_k(p,v)$ and so, $\sum_k p_k(p,v) = \frac{1}{2}2^{p+v} = 2^{p+v-1}$.

Now let v be even. Then, for even $k \neq 0$

$$\begin{aligned} v_k(p,v) - v_k(p,v-1) &= \sum_{0,k-p\leq 2r+1\leq v,k} \binom{v-1}{2r+1}\binom{p}{k-2r-1} \\ &- \sum_{0,k-p+1\leq 2r+1\leq v,k} \binom{v-1}{2r+1}\binom{p}{k-2r-1} \\ &= \sum_{0,k-p\leq 2r+1\leq v,k} [\binom{v}{2r+1} - \binom{v-1}{2r+1}]\binom{p}{k-2r-1} \\ &= \sum_{0,k-p\leq 2r+1\leq v,k} \binom{v-1}{2r}\binom{p}{k-2r-1} \\ &= \sum_{0,(k-1)-p\leq 2r\leq v-1,k-1} \binom{v}{2r}\binom{p-1}{(k-1)-2r} \\ &= p_{k-1}(p,v-1). \end{aligned}$$

Since $v-1$ is odd and $v_0(p,v)=0$, we get

$$\sum_k v_k(p,v) = \sum_{k>0} v_k(p,v-1) + \sum_{k>0} p_{k-1}(p,v-1) = 2^{p+v-2} + 2^{p+v-2} = 2^{p+v-1}$$

and consequently,

$$\sum_k p_k(p,v) = 2^{p+v-1}.$$

□

Theorem 4.8. Let $F = \mathrm{Diag}(a_1,...,a_d)$ and $G = \mathrm{Diag}(b_1,...,b_d)$ be diagonal integral matrices of non-zero determinant and such that $1 \leq a_1 \leq ... \leq a_d$. Let $f > 0$ denote the number of 1's in F, let v be the number of negative ones in $\{b_{f+1},...,b_d\}$ and let p be the number of ones in that same set. Then

(1) if $p = 0$ and $v = 0$, then
  (a) $K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \bigoplus_{0\leq k\leq d,\text{ even}} [\bigoplus_{I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}]$
  (b) $K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \bigoplus_{0\leq k\leq d,\text{ odd}} [\bigoplus_{I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}]$

(2) if $p = 0$ and $v > 0$, then
  (a) $K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \mathbb{Z}^{2^{v-1}-1} \oplus (\bigoplus_{0\leq k\leq d,\text{ odd}} [\bigoplus_{I\in\mathfrak{I}_k, b_I a_{I'}\neq 1} \mathbb{Z}_{1-b_I a_{I'}}]]$
  (b) $K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \mathbb{Z}^{2^{v-1}-1} \oplus (\bigoplus_{0\leq k\leq d,\text{ odd}} [\bigoplus_{I\in\mathfrak{I}_k, b_I a_{I'}\neq 1} \mathbb{Z}_{1-b_I a_{I'}}])$

(3) if $v = 0, p > 0$, we have
  (a) $K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \mathbb{Z}^{2^p} \oplus (\bigoplus_{0\leq k\leq d,\text{ even}} [\bigoplus_{I\in\mathfrak{I}_k, b_I a_{I'}\neq 1} \mathbb{Z}_{1-b_I a_{I'}}])$

(b) $K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \mathbb{Z}^{2^p} \oplus \big(\bigoplus_{0\leq k\leq d,\text{ odd}} [\bigoplus_{I\in\mathfrak{I}_k, b_I a_{I'}\neq 1} \mathbb{Z}_{1-b_I a_{I'}}]\big)$

(4) if $v \neq 0, p > 0$, then

(a) $K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \mathbb{Z}^{2^{p+v-1}} \oplus \big(\bigoplus_{0\leq k\leq d,\text{ even}} [\bigoplus_{I\in\mathfrak{I}_k, b_I a_{I'}\neq 1} \mathbb{Z}_{1-b_I a_{I'}}]\big)$

(b) $K_1(\mathcal{O}_{F,G}(\mathbb{T}^d)) = \mathbb{Z}^{2^{p+v-1}} \oplus \big(\bigoplus_{0\leq k\leq d,\text{ odd}} [\bigoplus_{I\in\mathfrak{I}_k, b_I a_{I'}\neq 1} \mathbb{Z}_{1-b_I a_{I'}}]\big)$

Proof. Begin by calculating

(1) $\bigoplus_{k,\text{ even(odd)}} \ker(1-C_k) = \bigoplus_{k,\text{ even(odd)}} \mathbb{Z}^{d_k}$

(2) $\bigoplus_{k,\text{ even(odd)}} \operatorname{coker}(1-C_k) = \bigoplus_{k,\text{ even(odd)}} \big(\bigoplus_{b_I a_{I'}\neq 1, I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}\big) \oplus \big(\bigoplus_{k,\text{ even(odd)}} \mathbb{Z}^{d_k}\big)$.

Thus,

$$\begin{aligned}
K_0(\mathcal{O}_{F,G}(\mathbb{T}^d)) &= \big(\bigoplus_{0\leq k\leq d,\text{ even}} \operatorname{coker}(1-C_k)\big) \oplus \big(\bigoplus_{0\leq k\leq d,\text{ odd}} \ker(1-C_k)\big) \\
&= \big(\bigoplus_{k,\text{ odd}} \mathbb{Z}^{d_k}\big) \oplus \big(\bigoplus_{k,\text{ even}} \big(\bigoplus_{b_I a_{I'}\neq 1, I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}\big) \oplus \big(\bigoplus_{k,\text{ even}} \mathbb{Z}^{d_k}\big) \\
&= \big(\bigoplus_{k} \mathbb{Z}^{d_k}\big) \oplus \big(\bigoplus_{k,\text{ even}} \big(\bigoplus_{b_I a_{I'}\neq 1, I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}\big) \\
&= \mathbb{Z}^{\sum_k d_k} \oplus \big(\bigoplus_{k,\text{ even}} \big(\bigoplus_{b_I a_{I'}\neq 1, I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}\big).
\end{aligned}$$

Similarly for K_1;

$$K_1(\mathcal{O}_{F,G}(\mathbb{T}^d) = \mathbb{Z}^{\sum_k d_k} \oplus \big(\bigoplus_{k,\text{ odd}} \big(\bigoplus_{b_I a_{I'}\neq 1, I\in\mathfrak{I}_k} \mathbb{Z}_{1-b_I a_{I'}}\big).$$

If $p = 0$ and $v = 0$, then there is no way to multiply to get 1. Hence, $1 - C_k$ is injective for all k and the result follows readily. If $p = 0$ and $v > 0$, then $d_k = 0$ for all odd k or $k > v$ and $d_k = \binom{v}{k}$ for all even $k \leq v$. Thus,

$$\sum_k d_k = \sum_{2\leq 2k\leq v} \binom{v}{2k} = \big(\sum_{0\leq 2k\leq v} \binom{v}{2k}\big) - 1 = 2^{v-1} - 1$$

and the rest follows.

If $p > 0$, then

$$d_k = \sum_{r\in\mathfrak{I}} \binom{v}{2r} \binom{p}{(k-d+f)-2r}$$

where $\mathfrak{I} = \{r \in \mathbb{N} \,\big|\, 0 \leq 2r \leq v, p-(k-d+f) \leq 2r \leq k-d+f\}$. Let $k' = k-d+f$ and then

$$d_k = p_{k'}(p, v)$$

where $p_{k'}(p,v)$ was defined in the above lemma and so,

$$\sum_k d_k = \sum_k p_k(p,v) = \begin{cases} 2^p & \text{if } v = 0 \\ 2^{p+v-1} & \text{if } v \neq 0 \end{cases}$$

and once again the rest follows.

$\square$

Remark 4.9. If f, the number of ones in F, is 0 then C_k is injective for $k = 0, ..., d-1$. It is then very simple to calculate the K-groups whether $\det G = 1$ or $\det G \neq 1$.

Corollary 4.10. If $F = n1_d, G = m1_d \in M_d(\mathbb{Z})$ where $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ are non-zero, then

(1) if either $n > 1$ or $|m| \neq 1$, then
  (a) $K_0(\mathcal{O}_{n,m}(\mathbb{T}^d)) = \bigoplus_{0 \leq k \leq d, \text{ even}} \mathbb{Z}_{1-n^{d-k}m^k}^{\binom{d}{k}}$
  (b) $K_1(\mathcal{O}_{n,m}(\mathbb{T}^d)) = \bigoplus_{0 \leq k \leq d, \text{ odd}} \mathbb{Z}_{1-n^{d-k}m^k}^{\binom{d}{k}}$

(2) if $n = m = 1$, then $K_0(\mathcal{O}_{1,1}(\mathbb{T}^d)) = K_1(\mathcal{O}_{1,1}(\mathbb{T}^d)) = \mathbb{Z}^{2^d}$

(3) if $n = 1$, $m = -1$, then $K_0(\mathcal{O}_{1,-1}(\mathbb{T}^d)) = K_1(\mathcal{O}_{1,-1}(\mathbb{T}^d)) = \mathbb{Z}^{2^{d-1}} \oplus \mathbb{Z}_2^{2^{d-1}}$

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